RESEARCH STATEMENT

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My research interests lie in algebraic number theory. Particularly, I like to study *Galois representations*, *modular forms*, and related objects to answer questions in number theory. Currently, I am working on the part of Ribet's method that constructs elements in certain Selmer groups using congruences between modular forms.

In what follows, I will convey the importance of Ribet's lemma and its generalizations in number theory, talk about my current and previous work, and conclude with future directions.

Ribet's lemma and generalizations. During my Ph.D., I worked on generalizing *Ribet's lemma* to symplectic groups (see Theorems 1 and 2). Let p be an odd prime. Given a two-dimensional irreducible p-adic representation that stabilizes a $\mathbb{Z}_p$-lattice and whose residual representation is given by two characters χ_1 and χ_2, Ribet's lemma constructs a nontrivial extension of χ_2 by χ_1 and χ_1 by χ_2 [**Rib76**, Proposition 2.1].

For a cuspidal eigenform g and any prime ℓ, there is an associated ℓ-adic Galois representation $\rho_{g,\ell}$ that encodes the Fourier coefficients of g [**Del71**]. For a prime p, if there is a congruence modulo p of the Fourier coefficients of g with the corresponding coefficients of an Eisenstein series, then $\rho_{g,p}$ satisfies the hypothesis of [**Rib76**, Proposition 2.1]. Using such a congruence, Ribet proves the converse to Herbrand's theorem [**Rib76**, Main Theorem 1.1].

Ribet's method of using congruences of modular forms to produce nontrivial elements in certain extension groups has many applications. For example, the Iwasawa theory main conjecture [**MW84, Wil90, Urb01, SU14**]. More recently, Dasgupta–Kakde and collaborators prove the Brumer–Stark conjecture by generalizing Ribet's lemma [**DK23, DKSW23a, DKSW23b**].

Bellaïche's theorem. Using the theory of Bruhat–Tits buildings, Bellaïche extends Ribet's lemma to GL_n [**Bel03**, Theorem 1.1]. Let V be an n-dimensional $\mathbb{Q}_p$-vector space, and let $\rho\colon G \to \mathrm{GL}_{\mathbb{Q}_p}(V)$ be an irreducible representation of a group G that admits a G-stable $\mathbb{Z}_p$-lattice. Let $\rho_1, \ldots, \rho_d$ be the distinct residual factors of ρ. The *extension graph* $\Gamma_\rho^{\mathrm{GL}}$ of ρ is the directed graph on $\rho_1, \ldots, \rho_d$, where an edge from ρ_i to ρ_j exists if there is a G-stable lattice Λ such that $\Lambda/p\Lambda$ admits a nontrivial extension of ρ_i by ρ_j as a subquotient. Bellaïche proves that the directed graph $\Gamma_\rho^{\mathrm{GL}}$ is strongly connected [**Bel03**, Theorem 1.1]. For applications and generalizations of Bellaïche's theorem, see [**BG06, BC04, BC14, BK21, OW24a, OW24b**].

Main theorem: two Jordan–Hölder factors. I prove the following generalization of [**Urb01**, Proposition 1.1] and [**Rib76**, Proposition 2.1].

Theorem 1. *Let $\rho\colon G \to \mathrm{GSp}_{2n}(\mathbb{Z}_p)$ be an irreducible symplectic representation, with similitude character λ, such that the semisimplification of the residual representation $\overline{\rho}$ is $\rho_1 \oplus \rho_2$. There is a basis of $\mathbb{Q}_p^{2n}$ with respect to which the residual representation $\overline{\rho}\colon G \to \mathrm{GSp}_{2n}(\mathbb{F}_p)$ is a nontrivial extension of ρ_2 by ρ_1 if and only if $\rho_2 \simeq \bar{\lambda} \otimes \rho_1^{-t}$.*

Symplectic extension graph and Ribet's lemma. In this section, I extend Theorem 1 in the spirit of Bellaïche by working with Bruhat–Tits buildings associated with symplectic groups (see Theorem 2). More specifically, I define the *symplectic extension graph* associated with an irreducible symplectic representation, where each edge corresponds to an extension realized by a stable *symplectic lattice*. This section ends with my main results and remarks on the methods used.

Let A be a ring such that 2 is invertible in A. A *symplectic A-module* is a pair (V,ϕ), where V is a finite free A-module equipped with a perfect A-bilinear pairing $\phi : V\times V \to A$ such that $\phi(v,v)=0$ for all $v\in V$. We denote by $\mathrm{GL}_A(V,\phi)$ the set of automorphisms $f : V\to V$ that preserves ϕ up to a scalar $\lambda(f)\in A^\times$.

Let (V,ϕ) be a symplectic $\mathbb{Q}_p$-vector space of dimension $2n$. A group representation $\rho\colon G\to \mathrm{GL}_{\mathbb{Q}_p}(V)$ is said to be *symplectic* if $\rho(G)$ lies in $\mathrm{GL}_{\mathbb{Q}_p}(V,\phi)$. The associated character $\lambda\circ\rho\colon G\to\mathbb{Q}_p^\times$ is called the *similitude character* of ρ and is denoted by λ_ρ. A $\mathbb{Z}_p$-lattice Λ of V is said to be *symplectic* if there exists a symplectic basis $\{w_1,\dots,w_n,v_1,\dots,v_n\}$ of V and integers $\{a_1,\dots,a_n,\mu\}$, such that Λ is the $\mathbb{Z}_p$-span of $\{p^{a_1}w_1,\dots,p^{a_n}w_n,p^{\mu-a_1}v_1,\dots,p^{\mu-a_n}v_n\}$.

Setup 1. Let $\rho\colon G\to \mathrm{GL}_{\mathbb{Q}_p}(V,\phi)$ be an irreducible symplectic representation that admits a G-stable symplectic lattice with absolutely irreducible residual factors $\rho_1,\dots,\rho_d$ that are pairwise nonisomorphic.

Definition 1. Let ρ be as in Setup 1. The *symplectic extension graph* $\Gamma_\rho^{\mathrm{GSp}}$ associated with ρ is the subgraph of $\Gamma_\rho^{\mathrm{GL}}$ where an edge from ρ_i to ρ_j exists if there is a G-stable symplectic lattice Λ such that $\Lambda/p\Lambda$ admits a nontrivial extension of ρ_i by ρ_j as a subquotient.

Our results. Let ρ be as in Setup 1, and let $I=\{1,\dots,d\}$. For $i\in I$, we let $\rho_i^\perp := \bar\lambda_\rho\otimes\rho_i^{-t}$, where ^{t} denotes the transpose. Since ρ stabilizes a $\mathbb{Z}_p$-lattice, we have $\lambda_\rho(G)\subseteq\mathbb{Z}_p^\times$. Let $\bar\lambda_\rho$ be the reduction modulo p of λ_ρ.

We make the following assumption:

(*) There exists $g_0\in G$ such that $\overline{\rho}(g_0)$ has distinct eigenvalues.

Theorem 2. *Let ρ be as in Setup 1 satisfying (*). The directed graph $\Gamma_\rho^{\mathrm{GSp}}$ is a skew-symmetric graph (in the sense of* [**GK96**]*) with respect to the transposition σ on I defined by $\rho_{\sigma(i)}\simeq\rho_i^\perp$ for all $i\in I$. Moreover, if $I_0(\rho) := \{i\in I \mid i=\sigma(i)\}=\varnothing$, then the directed graph $\Gamma_\rho^{\mathrm{GSp}}$ satisfies the following conditions.*

(a) *It has at most $d/2$ strongly connected components, with no edges between any two strongly connected components.*
(b) *For all $i\in I$, ρ_i and $\rho_{\sigma(i)}$ lie in the same strongly connected component.*
(c) *If $\Gamma_\rho^{\mathrm{GSp}}$ has $d/2$ connected components, then $\Gamma_\rho^{\mathrm{GSp}}$ is the disjoint union of strongly connected graphs with vertices $\{\rho_i,\rho_{\sigma(i)}\}_{i<\sigma(i)}$.*

We produce examples to show that Theorem 2 is an optimal result in this generality:

Theorem 3. *Given a positive integer n, and an integer $r\in\{1,\dots,n\}$, there exists a group G_r, and an irreducible symplectic representation $\rho\colon G_r\hookrightarrow \mathrm{GSp}_{2n}(\mathbb{Z}_p)$ such that*

(i) $I_0(\rho)=\varnothing$,
(ii) $\Gamma_\rho^{\mathrm{GSp}}$ *has r strongly connected components.*

Methods used. Let V be a finite dimensional $\mathbb{Q}_p$-vector space. Bellaïche uses the theory of Bruhat–Tits buildings associated with $\mathrm{GL}(V)$, generalizing a method of Serre, to prove [**Bel03**, Theorem 1.1]. If V has a symplectic form ϕ on it, then we prove Theorem 2 by working with the "smaller" building associated with $\mathrm{GL}(V,\phi)$.

The (Euclidean) *Bruhat–Tits building* associated with a p-adic linear algebraic group is a combinatorial and geometric object used to study the group and its representations. To study the compact subgroups of a complex Lie group, one constructs a CAT(0)-space on which the Lie group acts transitively. The p-adic analogue of such a CAT(0)-space is a Bruhat–Tits building. If $\mathbf{G}$ is a linear algebraic group, then one can realize the building associated with $\mathbf{G}(\mathbb{Q}_p)$ as a simplicial complex. For example, the Bruhat–Tits building associated with $\mathrm{GL}_2(\mathbb{Q}_p)$ is a $(p+1)$-regular infinite *tree*.

Let $\rho\colon G \to \mathrm{GL}(V, \phi)$ be an irreducible residually multiplicity free representation admitting a G-stable symplectic lattice. Under the assumption $I_0(\rho) = \varnothing$, we can move between G-stable symplectic vertices and prove the existence of such vertices that are "extremal" as ρ is irreducible. When $\dim V = 2$, the extremal vertices are the endpoints of the finite tree formed by the G-stable vertices. The extremal vertices realize directed edges in our graph $\Gamma_\rho^{\mathrm{GSp}}$. Given an integer $r \leq n$, we define a metric $\eta^{(r)}$ on I such that $G_r := \{(a_{ij}) \in \mathrm{GSp}_{2n}(\mathbb{Z}_p) \mid a_{ij} \in p^{\eta^{(r)}(i,j)}\mathbb{Z}_p\}$ with the inclusion $\rho\colon G_r \hookrightarrow \mathrm{GSp}_{2n}(\mathbb{Z}_p)$ satisfies the conclusions of Theorem 3.

Density results for sequences of big Galois representations. In this section, I talk about my past work on a zero density result for sequences of *big Galois representations* [**SS24**]. Galois representations play an important role in modern number theory. They serve as a bridge between the geometric world of varieties and the analytic world of modular forms. One well known instance of this is the modularity theorem proved by Wiles and others for elliptic curves over $\mathbb{Q}$, which was the final piece of puzzle in the proof of Fermat's last theorem [**Wil95, TW95, BCDT01**].

Galois representations arising in geometry, say associated with elliptic curves, are ramified at finitely many primes. But, infinitely ramified Galois representations do arise naturally in number theory [**Ram00**, Theorem 1]. The next question is how "big" is the set, possibly infinite, of ramified primes. In this direction, Khare–Rajan prove the set of ramified primes of a semisimple Galois representation has zero density [**KR01**, Theorem 1]. Khare generalizes this result to (trace) convergent sequences of Galois representations [**Kha03**, Proposition 1]. Building on the work of Saha [**Sah19**], Saha and I generalize [**Kha03**, Proposition 1] to big Galois representations.

Theorem 4. [**SS24**, Theorem 2.3] *Let $(\rho_i\colon G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{Z}_p[[X_1, \dots, X_m]]))_{i=1}^{\infty}$ be a convergent sequence of residually irreducible representations of the absolute Galois group $G_{\mathbb{Q}}$ of $\mathbb{Q}$. Then the set of primes where at least one of ρ_i ramifies has zero density.*

Future directions. In this section, I mention a few future directions I would like to work on in which our results, and their generalizations, might be useful.

Unitary analogue. As noted earlier, Theorem 2 is a generalization of [**Urb01**, Proposition 1.1]. Skinner–Urban prove a unitary $\mathrm{GU}(2,2)$-analogue of the symplectic GSp_4-result [**Urb01**, Proposition 1.1] in their proof of the Iwasawa main conjectures for GL_2 [**SU14**, §4].

Question 1. Can we obtain a unitary analogue of Theorem 2 for $\mathrm{GU}(m, n)$ for $n, m \geq 1$?

Methods analogous to ours should provide an answer to Question 1 when $n = m$. In other cases, the unitary Bruhat–Tits buildings are more subtle. The starting point in proving Question 1 is understanding these unitary Bruhat–Tits buildings.

Constructing symplectic extensions. The symplectic extension graph (see Definition 1) gives information about nontrivial extensions that are "symplectic" in some sense. In this direction, I would like to study a version of Ribet's lemma (generalizing Theorem 2) in the spirit of Bellaïche–Chenevier [**BC09**]. In particular, I want to pursue a symplectic version of [**BC09**, Theorem 1.5.5], which is a generalization of Ribet's lemma for *pseudocharacters* using the language of *generalized matrix algebras* (GMAs).

Let A be a henselian local ring with maximal ideal $\mathfrak{m}$, $\lambda : G \to A^\times$ be a character, and let $\tau : A[G] \to A[G]$ be the A-linear involution defined on G by $\tau(g) = \lambda(g)g^{-1}$. Let $T : A[G] \to A$ be an irreducible residually multiplicity free pseudocharacter such that $T \circ \tau = T$. Then the involutive A-algebra $(A[G]/\ker T, \tau)$ is a symplectic GMA in the sense of Moakher–Quast [**MQ23**, Definition 4.1].

QUESTION 2. Let $\rho_1, \ldots, \rho_d$ be absolutely irreducible representations such that $T \bmod \mathfrak{m} = \sum_{i=1}^{d} \mathrm{Tr}\rho_i$. Do there exist A-modules $\{A_{i,j}\}$ for $1 \le i, j \le d$ associated with the GMA structure for $A[G]/\ker T$, and a unipotent radical U of the parabolic subgroup determined by $\bar{T}$ with a G-action such that

$$\mathrm{Hom}(A_{i,j}, A/\mathfrak{m}) \hookrightarrow H^1(G, U_{k_i}/U_{k_j}),$$

where $1 \subset U_1 \subset \cdots \subset U_s = U$ is the central derived series for U?

When $d = 2$ and $\rho_2 = (\rho_1 \circ \tau)^t$, for $1 \le i \ne j \le 2$, let $U^{i,j}$ be the (abelian) unipotent radical of the Siegel parabolic of $\mathrm{GSp}_{2n}(A/\mathfrak{m})$ on which G-acts by conjugation by $\mathrm{diag}(\rho_i, \rho_j)$. Using [**BC09**, Theorem 1.8.6], we have some evidence for Question 2.

THEOREM 5. *If $d = 2$ and $\rho_2 = (\rho_1 \circ \tau)^t$, then we have an inclusion* $\mathrm{Hom}(A_{i,j}, A/\mathfrak{m}) \hookrightarrow H^1(G, U^{i,j})$ *for $1 \le i \ne j \le 2$.*

When $A = \mathbb{Z}_p$, Theorem 5 gives Theorem 1. We believe the structure theory of symplectic GMAs (proved in [**MQ23**]) should give an insight to approaching Question 2. Answering Question 2 should have many applications. We state one such in the next section.

Lifting symplectic representations mod p^2. To apply Ribet's method, one studies irreducible *deformations* (or *lifts*) of reducible representations. We would like to study irreducible lifts of $\overline{\rho}$ satisfying Setup 1. A first step in finding such symplectic lifts over $\mathbb{Z}_p$ is to find a $\mathbb{Z}/p^2\mathbb{Z}$ lift. When does such a lift exist? This is a symplectic analogue of [**KL24**, Question 1.1], which has been answered in certain cases [**Kha97, BÖ3, Bel12, MS26**].

QUESTION 3. Let A be a local ring with maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}^2 = 0$ and residue field $\mathbb{F}$. Let $\bar{T} : \mathbb{F}[G] \to \mathbb{F}$ be a pseudocharacter such that $\bar{T} \circ \tau = \bar{T}$. When does there exist an irreducible pseudocharacter $T : A[G] \to A$ lifting $\bar{T}$ such that $T \circ \tau = T$?

Let S be a finite set of primes in $\mathbb{Q}$ including ∞ and let $G = G_{\mathbb{Q},S}$ be the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside S. By answering Question 2, we believe that, if $\bar{T} = \mathrm{Tr}\rho_1 + \mathrm{Tr}\rho_2$ where $\rho_2 = \rho_1^\perp$, then Question 3 has an affirmative answer if and only if $H^1(G, U^{i,j}) \ne 0$ for $1 \le i \ne j \le 2$. This is a symplectic analogue of [**Bel12**, Proposition 2]. In the process of answering Question 3, a natural step is to generalize the results of [**Bel12**] and study the tangent space of the universal symplectic deformation ring associated with a symplectic pseudocharacter $\bar{T}$ as in Question 3.

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